

SIMILARITY, CODEPTH TWO BICOMODULES AND QF BIMODULES

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ABSTRACT. For any k -coalgebra C it is shown that similar quasi-finite C -comodules have strongly equivalent coendomorphism coalgebras; (the converse is in general not true). As an application we give a general result about codepth two coalgebra homomorphisms. Also a notion of codepth two bicomodule is introduced. The last section applies similarity to an endomorphism ring theorem for quasi-Frobenius (QF) bimodules and then to finite depth ring extensions. For QF extensions, we establish that left and right depth two are equivalent notions as well as a converse endomorphism theorem, and characterize depth three in terms of separability and depth two.

INTRODUCTION

For a ring R , two right R -modules M and N are *similar* [1] (H -equivalent in sense of Hirata) if M is a direct summand of $N^{(n)}$ and N is a direct summand of $M^{(m)}$ for some $m, n \in \mathbb{N}$. For example, M is similar to R_R if and only if M is a *progenerator*, i.e., M is a finitely generated projective generator. Hirata showed in [4] that similar R -modules $M \sim N$ have Morita equivalent endomorphism rings E_M and E_N ; whence isomorphic centers, $\text{End}_{E_M} M_R \cong \text{End}_{E_N} N_R$. (The converse is not true in general.) By taking R to be the enveloping ring of two rings, we extend this to a notion of similar bimodules.

For a k -coalgebra C , the notion of *ingenerator* comodule is introduced by Lin in [9] to characterize *strong equivalences* between comodule categories. A right C -comodule M is said to be an ingenerator if there are $m, n \in \mathbb{N}$ and $P, Q \in \mathcal{M}^C$ such that $M \oplus P \cong C^{(m)}$ and $C \oplus Q \cong M^{(n)}$. More generally, two right C -comodules M_C and N_C are called *similar* if there are $m, n \in \mathbb{N}$ and $P, Q \in \mathcal{M}^C$ such that $M \oplus P \cong N^{(m)}$ and $N \oplus Q \cong M^{(n)}$. Our first result in this note is to extend the result of Hirata to coalgebras. More precisely, if two quasi-finite C -comodules are similar, then their coendomorphism coalgebras are strongly equivalent in sense of Lin [9]. Using the notion of codepth two coalgebra homomorphism, introduced by the second author in [5], it is proved in Section 2, as an application, that any codepth two coalgebra homomorphisms $\varphi : C \rightarrow D$, such that C_D and $(C \square_D C)_D$ are

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quasi-finite comodules, have strongly equivalent coendomorphism coalgebras $e_{-D}(C)$ and $e_{-D}(C \square_D C)$. Also in Section 2 a notion of *codepth two bicomodule* is introduced, where it is noted that $\varphi : C \rightarrow D$ is a codepth two coalgebra homomorphism if and only if the ${}_D C_C$ is a codepth two bicomodule.

In the last section, using similar bimodules, we establish for QF bimodules an endomorphism ring theorem. In case of a QF ring extension, the tensor-square is similar with its endomorphism rings. This is applied to depth two and depth three QF extensions, showing that several results in [7] generalize from Frobenius to QF extensions.

1. DEFINITIONS AND SOME RESULTS

Throughout this paper k is a field and \mathcal{M}_k stands for the category of k -vector spaces. A basic reference for theory of coalgebras is, for example, [13]. A coalgebra over k is a k -space C together with two k -linear maps $\Delta : C \rightarrow C \otimes C$ (the unadorned tensor product is understood to be over k) and $\epsilon : C \rightarrow k$ such that $(1_C \otimes \Delta)\Delta = (\Delta \otimes 1_C)\Delta$ and $(1_C \otimes \epsilon)\Delta = (\epsilon \otimes 1_C)\Delta = 1_C$.

A right C -comodule is a k -space M with a k -map $\rho_M : M \rightarrow M \otimes C$ such that $(\rho_M \otimes 1_C)\rho_M = (1_M \otimes \Delta)\rho_M$ and $(1_C \otimes \epsilon)\rho_M = 1_M$. If M and N are C -comodules, a comodule map from M to N is a k -map $f : M \rightarrow N$ such that $(f \otimes 1)\rho_M = \rho_N f$. The k -space of all comodules maps from M to N is denoted by $\text{Com}_C(M, N)$ and \mathcal{M}^C denotes the category of right C -comodules. In the same way we can construct the category of left C -comodules ${}^C \mathcal{M}$.

It is well known that \mathcal{M}^C is an abelian category. In fact, \mathcal{M}^C is a locally finite Grothendieck category (generated by finite dimensional comodules). The fundamental properties of the categories of comodules can be found in several places, see e.g. [14, 3]. Let C be an arbitrary coalgebra, M a right C -comodule and N be a left C -comodule, the *cotensor product* $M \square_C N$ is the kernel of the k -map $\rho_M \otimes 1 - 1 \otimes \rho_N : M \otimes N \rightarrow M \otimes C \otimes N$. Following [3], the cotensor product is a left exact functor $\mathcal{M}^C \times^C \mathcal{M} \rightarrow \mathcal{M}_k$. Moreover, the mapping $m \otimes c \mapsto \epsilon(c)m$ and $c \otimes n \mapsto \epsilon(c)n$ yield a natural isomorphism $M \square_C C \cong M$ and $C \square_C N \cong N$. If C and D are two coalgebras, M is a (C, D) -bicomodule if $(1_C \otimes \rho^+)\rho^- = (\rho^- \otimes 1_D)\rho^+$ where $\rho^- : M \rightarrow C \otimes M$ and $\rho^+ : M \rightarrow M \otimes D$ are the structure maps of M . Moreover, if X is a right C -comodule, then the map $1_X \otimes \rho^+ : X \otimes M \rightarrow X \otimes M \otimes D$ define over $X \otimes M$ a structure of right D -comodule. In this case, $X \square_C M$ is a D -subcomodule of $X \otimes M$. This defines a left exact functor $-\square_C M : \mathcal{M}^C \rightarrow \mathcal{M}^D$ that preserves direct sums (see [14]).

Recall from [14] that a right C -comodule M is called *quasi-finite* if $\text{Com}_C(Y, M)$ is finite dimensional for every finite dimensional comodule $Y \in \mathcal{M}^C$. This is equivalent to the existence of a left adjoint $\text{h}_{-C}(M, -)$, called *co-hom* functor, to $- \otimes M$. If M is a (D, C) -bicomodule, the functor $\text{h}_{-C}(M, -) : \mathcal{M}^C \rightarrow \mathcal{M}^D$ becomes a left adjoint to the *cotensor product* functor $-\square_D M : \mathcal{M}^D \rightarrow \mathcal{M}^C$. If we assume that M_C is a quasi-finite comodule, then $e_{-C}(M) = \text{h}_{-C}(M, M)$ is a coalgebra, called *co-endorphism coalgebra* of M . Furthermore, M is a $(e_{-C}(M), C)$ -bicomodule via $\theta_M : M \rightarrow e_{-C}(M) \otimes M$, where $\theta : 1_{\mathcal{M}^C} \rightarrow \text{h}_{-C}(M, -)\square_D M$ denotes the unit of the adjunction.

Consider now two quasi-finite right C -comodules M_C and N_C . If we denote by $D_M = e_{-C}(M)$ and $D_N = e_{-C}(N)$ their coendomorphism coalgebras, then we can consider the diagram

$$(1) \quad \mathcal{M}^{D_M} \begin{array}{c} \xrightarrow{-\square_{D_M} M} \\ \xleftarrow{\text{h}_{-C}(M, -)} \end{array} \mathcal{M}^C \begin{array}{c} \xrightarrow{\text{h}_{-C}(N, -)} \\ \xleftarrow{-\square_{D_N} N} \end{array} \mathcal{M}^{D_N}$$

The composition of functors yield a pair of functors between the comodules categories over the coendomorphism coalgebras:

$$\mathbf{F} = \text{h}_{-C}(N, -\square_{D_M} M) : \mathcal{M}^{D_M} \rightleftarrows \mathcal{M}^{D_N} : \text{h}_{-C}(M, -\square_{D_N} N) = \mathbf{G},$$

where $\mathbf{F}(D_M) \cong \text{h}_{-C}(N, M)$ and $\mathbf{G}(D_N) \cong \text{h}_{-C}(M, N)$.

Following [14], a *Morita-Takeuchi context* $\Omega = (C, D; {}_C M_D, {}_D N_C; f, g)$ consists of coalgebras C and D , bicomodules ${}_C M_D$ and ${}_D N_C$ and bilinear maps $f : C \rightarrow M \square_D N$ and $g : D \rightarrow N \square_C M$ satisfying the following commutative diagrams:

$$\begin{array}{ccc} M & \xrightarrow{\simeq} & M \square_D D \\ \simeq \downarrow & & \downarrow I \square g \\ C \square_C M & \xrightarrow{f \square I} & M \square_D N \square_C M \end{array} \quad \begin{array}{ccc} N & \xrightarrow{\simeq} & N \square_C C \\ \simeq \downarrow & & \downarrow I \square f \\ D \square_D N & \xrightarrow{g \square I} & N \square_C M \square_D N \end{array}$$

The context is said to be *strict* if f and g are bilinear isomorphisms. In this case, the categories \mathcal{M}^C and \mathcal{M}^D are *equivalent* and we say that C is *Morita-Takeuchi equivalent* to D . The strict condition of the context is equivalent to say that the bicomodules M and N are injective cogenerators finitely cogenerated [14, Theorem 2.5].

For any quasi-finite right C -comodules M_C and N_C , it was proved in [8] that

$$(2) \quad \Gamma_{M,N}^C = (D_M, D_N; \text{h}_{-C}(N, M), \text{h}_{-C}(M, N); f, g)$$

is a Morita-Takeuchi context with bilinear maps

$$f : D_M \rightarrow \text{h}_{-C}(N, M) \square_{D_N} \text{h}_{-C}(M, N)$$

and

$$g : D_N \rightarrow h_{-C}(M, N) \square_{D_M} h_{-C}(N, M)$$

defined, respectively, by

$$(f \otimes 1)\theta_M = (1 \otimes \theta_N)\bar{\theta}_M$$

$$(g \otimes 1)\bar{\theta}_N = (1 \otimes \bar{\theta}_M)\theta_N$$

where θ and $\bar{\theta}$ are the units of the adjoint pair

$$(h_{-C}(M, -), -\square_{D_M} M)$$

and

$$(h_{-C}(N, -), -\square_{D_N} N),$$

respectively.

2. SIMILAR COMODULES AND A STRONG EQUIVALENCE

In this section we consider the definition of similar comodules and we prove that similar quasi-finite comodules have strongly equivalent coendomorphism coalgebras.

2.1. Definition. Let C be a k -coalgebra. Two right C -comodules M and N are *similar*, abbreviated $M \sim N$, if there are $m, n \in \mathbb{N}$ and C -comodules P and Q such that $M \oplus P \cong N^{(m)}$ and $N \oplus Q \cong M^{(n)}$.

It is easy to see that “ \sim ” defines an equivalence relation on the class \mathcal{M}^C of right C -comodules. Notice that if M_C is quasi-finite, the functor $h_{-C}(M, -)$ preserves comodules similar to M , as does the functor $-\square_{D_M} M$, where $D_M = e_{-C}(M)$ is the coendomorphism coalgebra of M . This comes from the fact that the co-hom and cotensor functors preserve direct sums. One obtains from [9] that a right C -comodule M is an ingenerator of \mathcal{M}^C if and only if M is similar to C_C as comodules. By transitivity of “ \sim ”, we note that if there are several ingenerators in \mathcal{M}^C , then all are similar. Therefore we can state a first result concerning similarity properties of comodules.

2.2. Lemma. *Assume that M and N are quasi-finite right C -comodules. If $M_C \sim N_C$, then $h_{-C}(M, N)$ and $h_{-C}(N, M)$ are ingenerators in \mathcal{M}^{D_M} and \mathcal{M}^{D_N} , respectively.*

Proof. Suppose that $M_C \sim N_C$. Since the functor $h_{-C}(M, -)$ (respectively, $h_{-C}(N, -)$) preserve similar comodules to M (respectively, to N), we deduce that $D_M = h_{-C}(M, M) \sim h_{-C}(M, N)$ and $D_N = h_{-C}(N, N) \sim h_{-C}(N, M)$, hence we obtain the lemma. \square

Let C and D be two k -coalgebras. Recall from [9] that C is *strongly equivalent* to D if the category \mathcal{M}^C is equivalent to the category \mathcal{M}^D via inverse equivalences

$$\mathbf{F} : \mathcal{M}^C \rightleftarrows \mathcal{M}^D : \mathbf{G},$$

such that $\mathbf{F}(C)$ is an ingenerator of \mathcal{M}^D and $\mathbf{G}(D)$ is an ingenerator of \mathcal{M}^C .

The theorem below answers our first aim in the affirmative.

2.3. Theorem. *Let C be a k -coalgebra and M_C and N_C be quasi-finite C -comodules. If M_C is similar to N_C , then D_M is strongly equivalent to D_N .*

Proof. Assume $M_C \sim N_C$. It follows from Lemma 2.2 that $\mathbf{h}_{-C}(M, N)$ and $\mathbf{h}_{-C}(N, M)$ are ingenerators of \mathcal{M}^{D_M} and \mathcal{M}^{D_N} , respectively. By [14, Theorem 2.5], the context $\Gamma_{M,N}^C$ in (2) is strict. So, D_M is Morita-Takeuchi equivalent to D_N . From diagram (1) the equivalence of categories is induced by the composition functors \mathbf{F} and \mathbf{G} . Moreover $\mathbf{F}(D_M) \sim D_N$ and $\mathbf{G}(D_N) \sim D_M$, which shows that the equivalence $\mathbf{F} : \mathcal{M}^{D_M} \rightleftarrows \mathcal{M}^{D_N} : \mathbf{G}$ is strong in sense of [9]. Hence the coendomorphism coalgebras are strongly equivalent. \square

2.4. Remark. The converse fails in general since if k is an algebraically closed field and we consider two non isomorphic simple comodules S and S' of \mathcal{M}^C , then $\mathbf{h}_{-C}(S, S) = \text{Com}_C(S, S)^* = k^* \cong k$ and $\mathbf{h}_{-C}(S', S') \cong k$. Thus D_S and $D_{S'}$ are strongly equivalent but S and S' cannot be similar.

2.5. Application to codepth two coalgebra homomorphisms.

Let $\varphi : C \rightarrow D$ be a homomorphism of coalgebras over a field. Then C has an induced (D, D) -bicomodule structure and any C -comodule becomes a D -comodule via the corestriction functor $(-)_\varphi : \mathcal{M}^C \rightarrow \mathcal{M}^D$ (see [14]). A well-known result in the theory of coalgebras is that the corestriction functor has the coinduction functor $-\square_D C : \mathcal{M}^D \rightarrow \mathcal{M}^C$ as right adjoint. Thus we have the adjoint couple of functors

$$(3) \quad \mathcal{M}^D \begin{array}{c} \xrightarrow{-\square_D C} \\ \xleftarrow{(-)_\varphi} \end{array} \mathcal{M}^C$$

A coalgebra homomorphism $\varphi : C \rightarrow D$ is called *left codepth two* [5, Definition 6.1] if the cotensor product $C \square_D C$ is isomorphic to a direct summand of a finite direct sum of C as (D, C) -bicomodules. Since C is in general isomorphic to a direct summand of $C \square_D C$ as (D, C) -bicomodules, we note that φ is a left codepth two coalgebra homomorphism if $C \square_D C$ and C are similar as (D, C) -bicomodules. Right codepth two coalgebra homomorphisms are similarly defined. Applying Theorem 2.3 to codepth two coalgebra homomorphism, we obtain:

Theorem. *Let $\varphi : C \rightarrow D$ be a codepth two coalgebra homomorphism. If C_D and $(C \square_D C)_D$ are quasi-finite right D -comodules, then the coendomorphism coalgebras $e_{-D}(C)$ and $e_{-D}(C \square_D C)$ are strongly equivalent.*

Consider now a (D, C) -bicomodule M with M_C quasi-finite. We have the adjoint pair

$$(\mathbf{h}_{-C}(M, -), -\square_D M).$$

A notion of codepth two bicomodule is the following.

Definition. A (D, C) -bicomodule M where M_C is quasi-finite is said to be *codepth two bicomodule* if $\mathbf{h}_{-C}(M, e_{-C}(M)\square_D M) \sim e_{-C}(M)$ as (C, D) -bicomodules.

Note that $\varphi : C \rightarrow D$ is a codepth two coalgebra homomorphism if and only if ${}_D C_C$ is a codepth two bicomodule.

3. APPLICATIONS OF SIMILARITY TO QF BIMODULES AND FINITE DEPTH EXTENSIONS

In this section, we recall some known notions and provide several new results concerning quasi-Frobenius bimodules and depth two extensions. For instance, we give the endomorphism ring theorem for QF-bimodules and several characterizations of finite projective depth two extensions in terms of similar bimodules.

3.1. On QF bimodules. For a unital bimodule ${}_B M_A$ over two unital rings B and A , we let $({}_B M)^*$ denote its left B -dual and $(M_A)^*$ its right A -dual. A unital ring homomorphism $\varphi : B \rightarrow A$ is referred to as a *ring extension* (of A over B , or A/B). We say (B, A) -bimodules M *divides* N , or $M \mid N$, if and only if $M \oplus P \cong N^{(n)}$ for some complementary (B, A) -bimodule P and direct sum power of N . Of course, $N \sim M$ if $M \mid N$ and $N \mid M$. Since the notion of division of modules may be formulated in terms of split exact sequences of the form $0 \rightarrow M \rightarrow N^{(n)} \rightarrow X \rightarrow 0$, similarity is clearly an equivalence relation, which is preserved by functors with the property of preserving finite direct sums.

Recall from [11] that φ is a left QF-extension if ${}_B A$ is finitely generated projective and $A \mid ({}_B A)^*$ as (A, B) -bimodules. Similarly, φ is a right QF-extension if A_B is finitely generated projective and $A \mid (A_B)^*$ as (B, A) -bimodules. We easily conclude that φ is a QF-extension (left and right extension) if and only if ${}_B A$ is finitely generated projective and $A \sim ({}_B A)^*$ as (A, B) -bimodules.

Recall also from [2] that a (B, A) -bimodule M is called *quasi-Frobenius bimodule*, or QF bimodule, if both ${}_B M$ and M_A are finitely generated projective and $({}_B M)^* \sim (M_A)^*$ as (A, B) -bimodules. It is easy to see that φ is a QF extension if and only if the natural bimodule ${}_A A_B$ is a QF-bimodule. In more detail, \Leftarrow is seen from $A \sim ({}_B A)^*$ as (A, B) -bimodules, so ${}_B A$ is finitely generated projective and $A \mid ({}_B A)^*$, so φ is by definition right QF. It follows that A_B is finitely generated projective. To the last similarity, we apply the functor $({}_B -)^*$ to obtain

$({}_B A)^* \sim A$ as (A, B) -bimodules, so $A \mid (A_B)^*$ and by definition φ is left QF.

Remark. In [2] the notion of QF Frobenius extension was extended to functors. In particular, φ is a QF extension if and only if the restriction of scalars functor φ_* is a QF functor. More generally, the bimodule ${}_B M_A$ is a QF bimodule if and only if the functor $M \otimes_A -$ is a QF functor [2, Proposition 3.5].

The calculus of similar bimodules lends itself to easy proofs of several results, e.g. of an endomorphism ring theorem for the QF property. For instance, if ${}_B M_A$ is a bimodule and $\lambda : B \rightarrow E = \text{End } M_A$ the left regular representation given by $\lambda_b(m) = bm$, then the endomorphism ring theorem for QF bimodules is stated as follows.

Proposition. *If ${}_B M_A$ is a QF bimodule, then $B \xrightarrow{\lambda} E$ is a QF extension.*

Proof. Since ${}_B M$ is finite projective, so is $({}_B M)^*_B$, and $(M_A)^*_B$ from the hypothesis that $(M_A)^* \mid ({}_B M)^*$ as right B -modules. Since M_A is finite projective and $E \cong M \otimes_A (M_A)^*$ (also as (E, B) -bimodules), it follows that E_B is finite projective.

It remains to show that $(E_B)^* \sim E$ as natural (B, E) -bimodules. We compute with the natural (B, E) -bimodule structures:

$$\begin{aligned} (E_B)^* &\cong \text{Hom}_{-B}(M \otimes_A (M_A)^*, B) \\ &\cong \text{Hom}_{-A}(M, \text{Hom}_{-B}((M_A)^*, B)) \\ &\sim \text{Hom}_{-A}(M, \text{Hom}_{-B}(({}_B M)^*, B)) \\ &\cong E \end{aligned}$$

since $(({}_B M)^*_B)^* \cong M$ follows for the reflexive module ${}_B M$. \square

3.2. On depth two extensions. Recall that a ring extension $B \rightarrow A$ is *right depth two*, or right D2, if the natural (A, B) -bimodules $A \otimes_B A$ and A itself are similar. A *left depth two* extension is defined oppositely: A/B is left D2 if $A^{\text{op}}/B^{\text{op}}$ is right D2. Also dual theorems for left D2 extensions may be deduced in this way.

As an example, an H-separable extension A/B is (left and right) D2, since its defining condition is that A and $A \otimes_B A$ be similar as natural (A, A) -bimodules [4]. Other examples are Hopf-Galois extensions, pseudo-Galois extensions, and faithfully flat projective algebras (The definition of depth two is sometimes extended in an straightforward way to include examples of infinite index subalgebras such as Hopf-Galois extensions with infinite dimensional Hopf algebra). A Hopf subalgebra of a semisimple Hopf \mathbb{C} -algebra is depth two precisely when it is normal. Just when a one-sided depth two ring extension is automatically two-sided is an interesting question of chirality [5] known to be the case also for Frobenius extensions: below we show that more generally QF extensions satisfy this property.

We give first a proposition with several characterizations of finite projective right D2 extensions in terms of similar bimodules.

3.3. Proposition. *Suppose A/B is a ring extension such that the natural modules ${}_B A$ and A_B are finite projective. Then the following are equivalent and characterize right D2 extension:*

- (a) $A \sim A \otimes_B A$ as (A, B) -bimodules;
- (b) $A \sim \text{End } {}_B A$ as (B, A) -bimodules;
- (c) $(A_B)^* \sim (A_B)^* \otimes_B (A_B)^*$ as (B, A) -bimodules.

Proof. Conditions (a) \Leftrightarrow (b) follow from [6, Proposition 3.8], which makes only use of ${}_B A$ finite projective in proving (b) \Rightarrow (a) (Note that any ring extension will satisfy $A | \text{End } {}_B A$ as (B, A) -bimodules since right multiplication $\rho : A \rightarrow \text{End } {}_B A$ is a split monic of (B, A) -bimodules, split by $f \mapsto f(1)$. The natural (A, A) -bimodule structure on $\text{End } {}_B A$ is given by $x \cdot f \cdot y = \rho_y \circ f \circ \rho_x$ for $x, y \in A$).

Condition (a) \Rightarrow (c) using A_B is finite projective: we note the isomorphism of natural (B, A) -bimodules,

$$(4) \quad (A_B)^* \otimes_B (A_B)^* \cong (A \otimes_B A_B)^*$$

via $\alpha \otimes_B \beta \mapsto (x \otimes_B y \mapsto \alpha(\beta(x)y))$, which follows from applications of [1, 20.11] and its dual. In this case, condition (c) follows from applying to (a) the functor $(-)_B^*$ from the category of (A, B) -bimodules into the category of (B, A) -bimodules.

The reverse implication, conditions (c) \Rightarrow (a) follows from the isomorphism (4), together with noting that A_B and therefore $A \otimes_B A_B$ are finite projective, and applying the functor $(B-)^*$ to condition (c). \square

3.4. Remark. The significance of the proposition is that the notion of depth two may be extended to functors as follows. Suppose $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor between abelian categories with left or right adjoint $G : \mathcal{D} \rightarrow \mathcal{C}$. Say that F is *depth two* if $FGF \sim F$, where similarity of functors is defined in a straightforward generalization of the two notions of similarity used above. If F is one of the three functors of restriction, induction or coinduction between module categories over rings B or A , then by the proposition any choice of F recovers the notion of depth two for (finite projective) ring extensions. For example, coinduction is tensoring by the dual in case the ring extension is finite projective.

We next show that QF extensions are two-sided depth two if one-sided. We first establish a lemma about QF extensions, which is well-known for Frobenius extensions (where isomorphism replaces similarity in the conclusion).

3.5. Lemma. *Let A/B be a QF ring extension. Then the natural (A, A) -bimodules $\text{End } {}_B A$, $A \otimes_B A$ and $\text{End } A_B$ are similar. Moreover, $A \otimes_B A \sim E$ as natural (E, A) -bimodules where $E = \text{End } A_B$.*

Proof. Assume that $A|B$ is a QF ring extension. Then $A \sim (A_B)^*$ as (B, A) -bimodules. Apply to this the functor $A \otimes_B -$ into the category of (E, A) -bimodules, obtaining $A \otimes_B A \sim \text{End } A_B$, since it follows from A_B finite projective that $\text{End } A_B \cong A \otimes_B (A_B)^*$. The rest of the proof is in the same vein. \square

3.6. Proposition. *Let A/B be a QF ring extension. Then A/B is left D2 \Leftrightarrow extension A/B is right D2.*

Proof. Suppose QF extension A/B is right D2, so that $A \sim \text{End } B A$ as (B, A) -bimodules by Proposition 3.3. By Lemma 3.5, $A \sim A \otimes_B A$ as (B, A) -bimodules, whence A/B is left D2. The converse follows from dualizing this argument. \square

With a bit more care, it may be shown that if a left QF extension is right D2, then it is left D2. Next we show a converse to the endomorphism ring theorem for the property D2. Let E denote $\text{End } A_B$ of a ring extension A/B .

3.7. Theorem. *Let A/B be a QF ring extension and A is a generator as a right B -module. If $A \xrightarrow{\lambda} E$ is D2, then A/B is D2.*

Proof. Since A_B is a progenerator, the rings E and B are Morita equivalent, with context bimodules ${}_E A_B$ and ${}_B (A_B)^*_E$. In particular, $(A_B)^* \otimes_E A \cong B$ as (B, B) -bimodules. Given the right D2 condition ${}_E E \otimes_A {}_E A \sim {}_E E A$, we make the substitution $E \sim A \otimes_B A$ as (E, A) -bimodules from the lemma. Thus, $A \otimes_B A \otimes_B A \sim A \otimes_B A$ as (E, A) -bimodules. Now apply the functor $(A_B)^* \otimes_E -$ to obtain $A \otimes_B A \sim A$ as (B, A) -bimodules; whence A/B is left D2. From Proposition 3.6, A/B is also right D2. \square

3.8. Depth three and more. From the point of view of finite depth and Galois correspondence, it is necessary to generalize depth two to a tower of three rings $C \subseteq B \subseteq A$, or more generally $C \rightarrow B \rightarrow A$ (denoting unital ring homomorphisms). Recall the tower $A/B/C$ is *right depth three*, or right D3, if $A \otimes_B A \sim A$ as natural (A, C) -bimodules. Thus, with $B = C$, we recover the notion of right D2 ring extension A/B . As an example of right D3 tower, let $A/B/C$ be the group algebras of a tower of groups $G > H > K$ over any commutative ground ring. Suppose the normal closure of K in G is contained in H : $K^G < H$. Then $A/B/C$ is (left and right) D3. (Similarly to Proposition 3.6, we may show that a QF tower of rings is two-sided D3 if one-sided.)

Depth three and more is originally an analytic notion for subfactors using the basic construction. Recall then from [7] that a ring extension A/B is *right depth three*, or right D3 extension, if the tower of rings $E/A/B$ is right D3, where E denotes $\text{End } A_B$ and the default mapping is as usual $A \xrightarrow{\lambda} E$. For example, when B and A are the group algebras of a finite subgroup pair $H < G$, the ring extension A/B is right D3

if H has a normal subgroup complement in G . As in subfactor theory, there is an embedding theorem for ring extensions that are depth three into ring extensions that are depth two; however, we provide a purely algebraic proof using only the QF property of ring extensions. Recall that a ring extension A/B is a *separable extension* if $A \otimes_B A$ contains a (separability) element $e = e^1 \otimes_B e^2$ (possibly suppressing a summation over simple tensors) such that $e^1 e^2 = 1_A$ and $ae = ea$ for all $a \in A$.

Theorem. *Suppose A/B is a separable QF-extension and $E = \text{End } A_B$. Then A/B is D3 \Leftrightarrow the composite extension E/B is D2.*

Proof. (\Rightarrow) This part of the proof does not require separability. We apply the bimodule similarity for the QF extension A/B , between its endomorphism ring and its tensor-square, ${}_A E_A \sim {}_A A \otimes_B A_A$. Tensoring by ${}_E E \otimes_A - \otimes_A E_A$, we obtain $E \otimes_A E \otimes_A E \sim E \otimes_B E$ as natural (E, A) -bimodules. Now tensor by ${}_E E \otimes_A -$ the right D3 condition $E \sim E \otimes_A E$ to obtain ${}_E E \otimes_A E_B \sim {}_E E \otimes_B E_B$. Again applying $E \sim E \otimes_A E$, we obtain ${}_E E \otimes_B E_B \sim {}_E E_B$. Thus E/B is right D2. Since it is a QF extension as well by Proposition 3.6, it is also left D2.

(\Leftarrow) This part of the proof does not require A/B be a QF extension. Since A/B is a separable extension, the natural (E, E) -epimorphism $E \otimes_B E \rightarrow E \otimes_A E$ splits via a mapping $x \otimes_A y \mapsto x e^1 \otimes_B e^2 y$ where $e = e^1 \otimes_B e^2$ denotes a separability element. Thus $E \otimes_A E$ divides $E \otimes_B E$ which is similar to E as (E, B) -bimodules. It follows that $E \otimes_A E$ divides E as (E, B) -bimodules, which is a sufficient condition for A/B to be right D3 extension (since E divides $E \otimes_A E$ via multiplication). \square

Recall that higher depth is defined by iterating the endomorphism ring construction. Thus a QF extension A/B is *depth n* if $E_{n-2}/E_{n-3}/B$ is a D3 tower, where $E_1 = E$, $E_0 = A$, and $E_{-1} = B$, and E_m is the right endomorphism ring of the extension $E_{m-2} \xrightarrow{\lambda} E_{m-1}$ defined inductively from $m \geq 2$. It is not hard to establish by similar means to those above that a depth n extension is also depth $n + 1$. From this and a tunneling lemma we may establish (by small modifications to the arguments in [7, section 8]) an embedding theorem for a depth n QF extension A/B ; that E_m/B is D2 for a sufficiently large $m \geq n - 2$.

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